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A combinatorial approach to coarse geometry ☆

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ABSTRACT

Using ideas from shape theory we embed the coarse category of metric spaces into the category of direct sequences of simplicial complexes with bonding maps being simplicial. Two direct sequences of simplicial complexes are equivalent if one of them can be transformed to the other by contiguous factorizations of bonding maps and by taking infinite subsequences. This embedding can be realized by either Rips complexes or analogs of Roe's anti-Čech approximations of spaces.

In this model coarse n-connectedness of $\mathcal{K} = \{K_1 \to K_2 \to \cdots\}$ means that for each k there is $m > k$ such that the bonding map from K_k to K_m induces trivial homomorphisms of all homotopy groups up to and including n.

The asymptotic dimension being at most n means that for each k there is $m > k$ such that the bonding map from K_k to K_m factors (up to contiguity) through an n-dimensional complex.

Property A of G. Yu is equivalent to the condition that for each k and for each $\epsilon > 0$ there is $m > k$ such that the bonding map from $|K_k|$ to $|K_m|$ has a contiguous approximation $g : |K_k| \to |K_m|$ which sends simplices of $|K_k|$ to sets of diameter at most ϵ.

1. Introduction

In homotopy theory the class with optimal properties consists of CW complexes. Other spaces are investigated by mapping CW complexes to them. That leads to the concept of the *singular complex* $\mathrm{Sin}(X)$ of a space X together with the projection $p : \mathrm{Sin}(X) \to X$ so that the following conditions are satisfied:

(1) Any continuous map $f : K \to X$, K a CW complex, lifts up to homotopy to a continuous map $g : K \to \mathrm{Sin}(X)$ and g is unique up to homotopy.
(2) p induces isomorphisms of all singular homology groups.

Thus $p : \mathrm{Sin}(X) \to X$ is a universal object among all continuous maps from CW complexes to X and $\mathrm{Sin}(X)$ represents all information about X from the point of view of the weak homotopy theory. A map $f : X \to Y$ of topological spaces is a *weak homotopy equivalence* if it induces a bijection $f_* : [K, X] \to [K, Y]$ of sets of homotopy classes for all CW complexes K (equivalently, it induces a homotopy equivalence from $\mathrm{Sin}(X)$ to $\mathrm{Sin}(Y)$).

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The shape theory (see [3] and [9]) represents the dual point of view: an arbitrary topological space X is investigated by mapping X to CW complexes K. A map $f: X \to Y$ of topological spaces is a *shape equivalence* if it induces a bijection $f^*: [Y, K] \to [X, K]$ of sets of homotopy classes for all CW complexes K.

In contrast to the weak homotopy theory there is no universal continuous map from X to a particular CW complex. Instead, each space X has the (inverse) *Čech system* $\{K_\alpha, [p_\alpha^\beta], A\}$ with projections $p_\alpha: X \to K_\alpha$ which reflects the shape of X in the following sense:

a. For any continuous map $f: X \to K$ to a CW complex K there is $\alpha \in A$ and $g: K_\alpha \to K$ such that f is homotopic to $g \circ p_\alpha$.
b. Given $\alpha \in A$ and given continuous maps $g, h: K_\alpha \to K$ with $g \circ p_\alpha$ homotopic to $h \circ p_\alpha$ there is $\beta \geqslant \alpha$ so that $g \circ p_\alpha^\beta$ is homotopic to $h \circ p_\alpha^\beta$.

The Čech system of X consists of geometric realizations of nerves of numerable open coverings of X and the bonding maps are simplicial maps induced by functions between covers that reflect one of them being a refinement of the other.

In this paper we will show that the coarse category of metric spaces (see [6] and [7]) is dual to shape category in the following sense: for each X we consider the set of equivalence classes $C(K, X)$ of functions $f: K \to X$ from a simplicial complex K to X. It is required that the family $\{f(\Delta)\}_{\Delta \in K}$ is uniformly bounded (we call such functions *bornologous*) and f and g are equivalent if the family $\{f(\Delta) \cup g(\Delta)\}_{\Delta \in K}$ is uniformly bounded. Each metric space has a coarse Čech system $\{K_m\}$ consisting of a direct system of simplicial complexes and simplicial maps. This system is universal among all bornologous functions from simplicial complexes to X.

We show asymptotic dimension is dual to the shape dimension and coarse connectivity is dual to shape connectivity.

Convention: The metrics considered in this paper may attain infinite values. As explained in [2], the primary advantage of such metrics is the ability of constructing disjoint unions of metric spaces (see the proof of Proposition 2.2).

We will make a careful distinction between graphs (or simplicial complexes) and their geometric realization. Usually we will focus on the set of vertices $X = G^{(0)}$ of a graph G (or a simplicial complex K) and we may denote G (or K) by X in the absence of other graph (or simplicial complex) structures on X. By *geometric realization* $|G|$ (or $|K|$) we mean $G^{(0)}$ and the union of all geometric edges (geometric simplices) induced by edges in G (simplices in K). Thus, given a simplex $\Delta = [x_0, \ldots, x_n]$ (a finite subset of X), its geometric realization $|\Delta|$ is the set of all formal linear combinations $\sum_{i=0}^n t_i \cdot x_i$, where $t_i \geqslant 0$ and $\sum_{i=0}^n t_i = 1$.

Given a set X we can put a graph structure G on it by specifying all the edges (example: the Cayley graph of a finitely generated group). A graph structure G on X leads to the *graph metric* d_G on X as follows (notice the advantage of using metrics with infinite values):

(1) $d_G(v, w) = 0$ if and only if $v = w$,
(2) $0 < d_G(v, w) \leqslant n$ if and only if $v \neq w$ and there is a chain $v_0 = v, \ldots, v_n = w$ such that $[v_i, v_{i+1}]$ is an edge in G for each $0 \leqslant i < n$. $d_G(v, w) = k > 0$ if $d_G(v, w) \leqslant k$ and $d_G(v, w) \leqslant k - 1$ is false,
(3) $d_G(v, w) = \infty$ otherwise.

Conversely, given a metric d_X on X, we can put several graph structures on X:

Definition 1.1. Given a metric space (X, d_X) and $t > 0$, the *Rips graph* $\mathrm{RipsG}_t(X)$ consists of edges $[x, y]$ such that $d_X(x, y) \leqslant t$.

Observe $\mathrm{RipsG}_n(\mathrm{RipsG}_m(X, d_X)) = \mathrm{RipsG}_{m \cdot n}(X, d_X)$ for (X, d_X) geodesic and m, n positive integers.

Notice that $\pi_t: \mathrm{RipsG}_t(X) \to X$ induced by the identity function is t-Lipschitz. Moreover, t-Lipschitz maps from a graph V to X are in one-to-one correspondence with short maps from V to $\mathrm{RipsG}_t(X)$ (following Gromov by *short maps* we mean Lipschitz maps with the Lipschitz constant one).

Notice that a function $f: G \to H$ between graphs is short if and only if for every edge $[x, y]$ of G, $[f(x), f(y)]$ is an edge of H.

Observe that the metric d_X of a metric space (X, d_X) is induced by some graph structure on X if and only if d_X is integer valued and X is 1-geodesic. A metric space (X, d_X) is *t-geodesic* if for every two points x and y there is a t-chain $x_0 = x, \ldots, x_k = y$ (that means $d_X(x_i, x_{i+1}) \leqslant t$ for all $0 \leqslant i \leqslant k - 1$) such that $d_X(x, y) = \sum_{i=0}^{k-1} d_X(x_i, x_{i+1})$. Furthermore the graph metric d_G on G can be extended to a geodesic metric on the geometric realization $|G|$ of G.

One can generalize the concept of Rips graphs from metric spaces to arbitrary sets provided a cover of the set is given:

Definition 1.2. Given a set X and a cover $\mathcal{U}$ of X, the *Rips graph* $\mathrm{RipsG}_{\mathcal{U}}(X)$ consists of edges $[x, y]$ such that there is an element U of $\mathcal{U}$ containing x and y.

Notice $\mathrm{RipsG}_t(X) \subset \mathrm{RipsG}_{\mathcal{U}}(X) \subset \mathrm{RipsG}_{2t}(X)$, where $\mathcal{U}$ is the cover of X by closed t-balls.

2. The coarse category

In this section we introduce the coarse category of metric spaces in a way different (but equivalent) to that of Roe [11]. It is similar to the intuitive way of introducing asymptotic cones of metric spaces: one looks at a given metric space from farther and farther away. In our case the basic idea is to look at two functions from farther and farther away.

From the large scale point of view two functions $f, g : X \to (Y, d_Y)$ should be considered indistinguishable if they are within finite distance from each other (i.e., $\sup_{x \in X} d_Y(f(x), g(x)) < \infty$). Therefore it makes sense to consider the set of equivalence classes of functions from X to (Y, d_Y) with respect to this relation: we say that the f and g are *large scale equivalent* or *ls-equivalent* (and denote this by $f \sim_{ls} g$) if and only if $\sup_{x \in X} d_Y(f(x), g(x)) < \infty$. For f and g *ls*-equivalent and $C > \sup_{x \in X} d_Y(f(x), g(x))$ we say that f and g are *C-close*. Note that a metric on a domain space is not required for the definition of *ls*-equivalent maps. We denote by $[X, Y]_{ls}$ the set of *ls*-equivalence classes of functions from X to (Y, d_Y).

The next logical step, from the categorical point of view, is to consider functions that preserve the relation $\sim_{ls}$. Obviously, $f \sim_{ls} g$ implies $f \circ \alpha \sim_{ls} g \circ \alpha$ for any $\alpha : (Z, d_Z) \to (X, d_X)$, so the following definition addresses the crux of the matter:

Definition 2.1. A function $\alpha : (X, d_X) \to (Y, d_Y)$ of metric spaces is *bornologous* (or *large scale uniform*) if $f \sim_{ls} g$, $f, g : (Z, d_Z) \to (X, d_X)$, implies $\alpha \circ f \sim_{ls} \alpha \circ g$.

Clearly, α bornologous and $\alpha \sim_{ls} \beta$ implies β bornologous. Let us show that our definition of a function being bornologous is equivalent to that of Roe [11]:

Proposition 2.2. *If $\alpha : (X, d_X) \to (Y, d_Y)$ is a function of metric spaces, then the following conditions are equivalent*:

a. *α is bornologous.*
b. *For any $t > 0$ there is $s > 0$ such that $d_X(x, y) < t$ implies $d_Y(\alpha(x), \alpha(y)) < s$.*

Proof. Suppose α is bornologous. For $t > 0$ define a metric space $X_t = \coprod_{x \in X} B(x, t) \times \{x\}$ with metric

$$d_t\big((x_1, y_1), (x_2, y_2)\big) = \begin{cases} d_X(x_1, x_2), & y_1 = y_2, \\ \infty, & y_1 \neq y_2. \end{cases}$$

Maps $f, g : X_t \to X$ defined as $f(x, y) = x$ and $g(x, y) = y$ are at distance at most t, hence $f \sim_{ls} g$. Because α is bornologous, $\alpha \circ f \sim_{ls} \alpha \circ g$ which means that there exists $s > 0$ such that $d_X((\alpha \circ f)(x, y), (\alpha \circ g)(x, y)) < s$. Let $d_X(x, y) < t$, then $(x, y) \in X_t$. Because $(\alpha \circ f)(x, y) = \alpha(x)$ and $(\alpha \circ g)(x, y) = \alpha(y)$, the distance $d_Y(\alpha(x), \alpha(y)) < s$.

Suppose that for any $t > 0$ there is $s > 0$ such that $d_X(x, y) < t$ implies $d_Y(\alpha(x), \alpha(y)) < s$. Let $f, g : (Z, d_Z) \to (X, d_X)$ satisfy $\sup_{z \in Z} d_X(f(z), g(z)) = t < \infty$. Let s be as above. Then $d_X(\alpha(f(z)), \alpha(g(z))) < s$, hence $\alpha \circ f \sim_{ls} \alpha \circ g$. $\square$

Definition 2.3. A bornologous function $\alpha : (X, d_X) \to (Y, d_Y)$ of metric spaces is a *large scale isomorphism* if it induces a bijection $\alpha_* : [Z, X]_{ls} \to [Z, Y]_{ls}$ for all metric spaces (Z, d_Z).

Notice that every large scale isomorphism α is a proper map. Let $A \subset Y$ be a bounded subset and let $i, c : \alpha^{-1}(A) \to \alpha^{-1}(A) \subseteq X$ denote the inclusion and a constant map, respectively. The compositions $\alpha \circ i$ and $\alpha \circ c$ are *ls*-equivalent, hence i and c are *ls*-equivalent as well. Consequently $\alpha^{-1}(A)$ is bounded as $c(\alpha^{-1}(A))$ is a singleton.

As in the case of bornologous functions, our definition of large scale isomorphisms is equivalent to the definition of coarse equivalence in [11]:

Proposition 2.4. *If $\alpha : (X, d_X) \to (Y, d_Y)$ is a bornologous function of metric spaces, then the following conditions are equivalent*:

a. *α is a large scale isomorphism.*
b. *There is a proper bornologous function $\beta : (Y, d_Y) \to (X, d_X)$ such that $\alpha \circ \beta \sim_{ls} id_Y$ and $\beta \circ \alpha \sim_{ls} id_X$.*
c. *There is a bornologous function $\beta : (Y, d_Y) \to (X, d_X)$ such that $\alpha \circ \beta \sim_{ls} id_Y$ and $\beta \circ \alpha \sim_{ls} id_X$.*

Proof. (a) $\Rightarrow$ (b). Suppose α is a large scale isomorphism. The map $\alpha_* : [Y, X]_{ls} \to [Y, Y]_{ls}$ is a bijection, hence there exists a map $\beta : Y \to X$ such that $\alpha \circ \beta \sim_{ls} id_Y$. Because $\alpha \circ \beta \circ \alpha \sim_{ls} \alpha \circ id_X$ and $\alpha_* : [X, X]_{ls} \to [X, Y]_{ls}$ is a bijection, also $\beta \circ \alpha \sim_{ls} id_X$. Let us show that β is bornologous. Let $f, g : Z \to Y$ be maps between metric spaces and $f \sim_{ls} g$. Because $\alpha \circ \beta$ is bornologous, $(\alpha \circ \beta) \circ f \sim_{ls} (\alpha \circ \beta) \circ g$. Because $\alpha_* : [Z, X]_{ls} \to [Z, Y]_{ls}$ is a bijection, $\beta \circ f \sim_{ls} \beta \circ g$. Let us show that β is proper. Let $A \subset X$ be a bounded subset. Because α is bornologous, $\alpha(A)$ is bounded as well. On the other hand $\alpha \circ \beta \sim_{ls} id_Y$, hence $\beta^{-1}(A)$ is bounded as $(\alpha \circ \beta)(\beta^{-1}(A)) = \alpha(A)$ is bounded.

(c) $\Rightarrow$ (a). Suppose there is a bornologous function $\beta : (Y, d_Y) \to (X, d_X)$ such that $\alpha \circ \beta \sim_{ls} id_Y$ and $\beta \circ \alpha \sim_{ls} id_X$. Let Z be a metric space and $f, g : Z \to X$ such that $\alpha \circ f \sim_{ls} \alpha \circ g$. Then $f \sim_{ls} \beta \circ \alpha \circ f \sim_{ls} \beta \circ \alpha \circ g \sim_{ls} g$, hence $\alpha_* : [Z, X]_{ls} \to [Z, Y]_{ls}$ is injective. Let $h : Z \to Y$ be a map. Then $\alpha \circ (\beta \circ h) \sim_{ls} h$, so α_* is surjective. $\square$

In view of Proposition 2.2 the simplest bornologous functions are Lipschitz functions. Conversely, it is easy to deduce from Proposition 2.2 that every bornologous function defined on a t-geodesic space is Lipschitz.

The following result shows that Lipschitz maps from graphs are of primary interest in large scale geometry.

Theorem 2.5. *Let $f:(X,d_X)\to(Y,d_Y)$ be a function of metric spaces.*

a. *The map f is bornologous if and only if for every Lipschitz function $g:(G,d_G)\to X$, G any graph, $f\circ g$ is Lipschitz.*
b. *Suppose the map f is bornologous. Then it is a large scale isomorphism if and only if every Lipschitz function $h:G\to Y$, G any graph, lifts (up to large scale equivalence) to a Lipschitz map to X and the lift is unique up to large scale equivalence.*

Proof. (a). Let a map $f:(X,d_X)\to(Y,d_Y)$ be bornologous and $g:(G,d_G)\to X$ be a Lipschitz map with Lipschitz constant t. There exists $s>0$ such that $d_X(x,y)<t$ implies $d_Y(f(x),f(y))<s$. Let $a,b\in G$ and $d_G(a,b)=n$, then there exit a sequence $a_0,\dots,a_n\in G$ such that $a_0=a$, $a_n=b$, and $d_G(a_{i-1},a_i)=1$ for $i=1,\dots,n$. For $i=1,\dots,n$ the distance $d_X(g(a_{i-1}),g(a_i))<t$, hence $d_Y(fg(a_{i-1}),fg(a_i))<s$. Therefore

$$d_Y\big(fg(a),fg(b)\big)\leqslant\sum_{i=1}^{n}d_Y\big(fg(a_{i-1}),fg(a_i)\big)\leqslant\sum_{i=1}^{n}s=sd_G(a,b)$$

and the map $f\circ g$ is Lipschitz.

Suppose that for every Lipschitz function $g:(G,d_G)\to X$, G any graph, $f\circ g$ is Lipschitz. Let us prove that f is bornologous. Let $t>0$. The map $\pi_t:G=\mathrm{RipsG}_t(X)\to X$ is t-Lipschitz, hence the map $f\circ\pi_t$ is L-Lipschitz for some L. If $d_X(x,y)<t$ and $x\neq y$, then

$$d_Y\big(f(x),f(y)\big)=d_Y\big(f\pi_t(x),f\pi_t(y)\big)\leqslant Ld_G(x,y)=L,$$

so f is bornologous.

(b). Let f be a large scale isomorphism. There exists a bornologous function $g:Y\to X$, such that $f\circ g\sim_{ls} id_Y$ and $g\circ f\sim_{ls} id_X$. Let $h:G\to Y$ be a Lipschitz map. Let $\tilde h=g\circ h$ which is Lipschitz by (a). Then $f\circ\tilde h=f\circ g\circ h\sim_{ls} h$, so $\tilde h$ is a lift up to large scale equivalence of the map h. Suppose $h':G\to X$ be a Lipschitz map such that $f\circ h'\sim_{ls} h$. Then $\tilde h=g\circ h\sim_{ls} g\circ f\circ h'\sim_{ls} h'$, hence a lift is unique.

Suppose every Lipschitz function $h:G\to Y$, G any graph, lifts (up to large scale equivalence) to X and the lift is unique up to large scale equivalence. Consider lifts $h_t:\mathrm{RipsG}_t(Y)\to X$ of the projections $\pi_t:\mathrm{RipsG}_t(Y)\to Y$ for all $t>0$. If $s>t$, then both h_s and h_t can be considered as lifts of $\pi_t:\mathrm{RipsG}_t(Y)\to Y$, so they are ls-equivalent. Notice all $h_t:Y\to X$ are bornologous. Indeed, if $d_Y(x,y)<s$ and $s>t$, then $d_X(h_s(x),h_s(y))\leqslant L_s$, where L_s is the Lipschitz constant of h_s. Since there is $c>0$ such that $d_X(h_s(z),h_t(z))<c$ for all $z\in Y$, $d_X(h_t(x),h_t(y))\leqslant L_s+2c$.

For all $s>0$ the composition $f\circ h_s$ is ls-equivalent to $\pi_s=id_Y$ (recall that the metric on the domain is redundant when considering ls-equivalence between maps). It remains to show that $h_s\circ f\sim_{ls} id_X$ for some $s>0$. Choose $s>0$ such that $d_X(x,y)<1$ implies $d_Y(f(x),f(y))<s$ and notice $f:\mathrm{RipsG}_1(X)\to\mathrm{RipsG}_s(Y)$ is short. Then $h_s\circ f:\mathrm{RipsG}_1(X)\to X$ is a Lipschitz lift of $\pi_s\circ f:\mathrm{RipsG}_1(X)\to Y$. By the uniqueness of lift, $h_s\circ f\sim_{ls} id_X$ since the natural map $\mathrm{RipsG}_1(X)\to X$ is a lift of $\pi_s\circ f$ as well. □

3. Coarse graphs

Definition 3.1. Given a graph G, by $A(G)$ we mean the graph with the same set of vertices as G but the set of edges is increased by adding all $[v,w]$ such that $d_G(v,w)=2$.

In other words, $A(G)$ equals $\mathrm{RipsG}_t(G,d_G)$ for any $2<t<3$.

Notice the identity function $G\to A(G)$ is short and bi-Lipschitz for any graph G.

Definition 3.2. A *coarse graph* is a direct sequence $\{V_1\to V_2\to\cdots\}$ of graphs V_n and short maps $i_{n,m}:V_n\to V_m$ for all $n\leqslant m$ such that

(1) $i_{n,n}=id$ for all $n\geqslant 1$,
(2) $i_{n,k}=i_{m,k}\circ i_{n,m}$ for all $n\leqslant m\leqslant k$,
(3) for every $n\geqslant 1$ there is $m>n$ so that $i_{n,m}:A(V_n)\to V_m$ is short.

Definition 3.3. A *coarse graph of a metric space* (X, d_X) is a coarse graph $\{V_1 \to V_2 \to \cdots\}$ together with Lipschitz maps $p_n : V_n \to X$

$$\begin{tikzcd}
V_1 \arrow[r, "i_{1,2}"] \arrow[d, "p_1"'] & V_2 \arrow[r, "i_{2,3}"] \arrow[dl, "p_2"'] & V_3 \arrow[r, "i_{3,4}"] \arrow[dll, "p_3"] & \cdots \\
X & & &
\end{tikzcd}$$

for all n such that the following conditions are satisfied:

a. p_n is large scale equivalent to $p_{n+1} \circ i_{n,n+1}$ for each $n \geqslant 1$.
b. For any Lipschitz map $g : V \to X$ from a graph V there is $n \geqslant 1$ and a short map $g' : V \to V_n$ such that $p_n \circ g'$ is large scale equivalent to g.
c. If $g, h : V \to V_n$ are two short maps from a graph V such that $p_n \circ g$ is large scale equivalent to $p_n \circ h$, then there is $m > n$ such that $i_{n,m} \circ g$ is large scale equivalent to $i_{n,m} \circ h$.

Traditionally, a *Cayley graph* of a finitely generated group G is defined to be $\Gamma(G, S)$, where S is a symmetric finite set of generators of G not containing the neutral element 1_G. Its set of vertices is G and edges of $\Gamma(G, S)$ are precisely of the form $[g, g \cdot s]$, $g \in G$ and $s \in S$. However, one can easily generalize the concept of Cayley graphs to arbitrary groups G and arbitrary finite subsets S of $G \setminus \{1_G\}$ (actually, S may be infinite but we do not know any application of such graphs).

It is known that arbitrary countable group G has a proper left invariant metric d_G (see [12] or [13]) and any two such metrics are large scale equivalent. Our next result is a variant of that fact.

Proposition 3.4. *Suppose d_G is a left invariant proper metric on a group G. If S_n is an increasing sequence of finite symmetric subsets of G such that $G \setminus \{1_G\} = \bigcup_{n=1}^{\infty} S_n$, then the sequence $\{\Gamma(G, S_1) \to \Gamma(G, S_2) \to \cdots\}$ together with projections $\pi_i : \Gamma(G, S_i) \to G$ forms a coarse graph of (G, d_G).*

Proof. Obviously, maps $i_{n,m} : \Gamma(G, S_n) \to \Gamma(G, S_m)$ are identities on vertices and are short for $n \leqslant m$. For any n there is $m \geqslant n$ such that for every $s_n \in S_n$, $(s_n \cdot S_n) \setminus \{1_G\} \subset S_m$ in which case $i_{n,m} : A(\Gamma(G, S_n)) \to \Gamma(G, S_m)$ is short. It is enough to check (b) of Definition 3.3 for short maps. If $g : V \to G$ is a short map from a graph V to G, we pick S_n containing the punctured ball $B(1_G, 2) \setminus \{1_G\}$ at 1_G of radius 2 (as d_G is proper, that ball is finite). Now g considered as a map from V to $\Gamma(G, S_n)$ is short. Indeed, if $[v, w]$ is an edge in V, then $d_G(g(v), g(w)) \leqslant d_V(v, w) = 1$, so $d_G(1_G, g(v)^{-1} \cdot g(w)) \leqslant 1$ and $s = g(v)^{-1} \cdot g(w) \in S_n$. Therefore $[g(v), g(w)]$ is an edge in $\Gamma(G, S_n)$ proving $g : V \to \Gamma(G, S_n)$ is short.

Suppose $g, h : V \to \Gamma(G, S_n)$ are two short maps so that $\pi_n \circ g \sim_{ls} \pi_n \circ h$. There is $M > 0$ such that $d_G(g(v), h(v)) < M$ for all vertices v of V. Choose $m > n$ with the property that S_m contains the punctured ball $B(1_G, M) \setminus \{1_G\}$. As above, we can show $[g(v), h(v)]$ is an edge in $\Gamma(G, S_m)$ for any vertex v of V. Thus $i_{n,m} \circ g \sim_{ls} i_{n,m} \circ h$. $\square$

Example 3.5. The following example generalizes Rips graphs. Suppose (X, d_X) is a metric space and $\mathcal{U}_n$ is a sequence of uniformly bounded covers of X such that $\mathcal{U}_n$ is a refinement of $\mathcal{U}_{n+1}$ and Lebesgue numbers $L(\mathcal{U}_n)$ of $\mathcal{U}_n$ form a sequence diverging to infinity. The *Rips coarse graph* of X with respect to the sequence $\mathcal{U}_n$ is $\{\mathrm{RipsG}_{\mathcal{U}_1}(X) \to \mathrm{RipsG}_{\mathcal{U}_2}(X) \to \cdots\}$ together with projections $\pi_i : \mathrm{RipsG}_{\mathcal{U}_i}(X) \to X$ where

(1) maps π_i are induced by the identity id_X;
(2) maps $i_{n,m} : \mathrm{RipsG}_{\mathcal{U}_n}(X) \to \mathrm{RipsG}_{\mathcal{U}_m}(X)$ are identities on vertices.

Proposition 3.6. *If $\mathcal{U}_n$ is a sequence of uniformly bounded covers of X such that $\mathcal{U}_n$ is a refinement of $\mathcal{U}_{n+1}$ and Lebesgue numbers form a sequence $L(\mathcal{U}_n) \to \infty$ as $n \to \infty$, then the sequence $\{\mathrm{RipsG}_{\mathcal{U}_1}(X) \to \mathrm{RipsG}_{\mathcal{U}_2}(X) \to \cdots\}$ together with projections $\pi_i : \mathrm{RipsG}_{\mathcal{U}_i}(X) \to X$ forms a coarse graph of (X, d_X).*

Proof. We first prove that $\{\mathrm{RipsG}_{\mathcal{U}_1}(X) \to \mathrm{RipsG}_{\mathcal{U}_2}(X) \to \cdots\}$ is a coarse graph. Because $\mathcal{U}_n$ is a refinement of $\mathcal{U}_{n+1}$ all identity maps $i_{n,m}$ are short. Furthermore because the edges of $A(\mathrm{RipsG}_{\mathcal{U}_n}(X))$ are a subset of edges of $\mathrm{RipsG}_{\mathcal{U}_m}(X)$ provided $L(\mathcal{U}_m) \geqslant d$ (where $\mathcal{U}_n$ is d-bounded) the map $i_{n,m} : A(\mathrm{RipsG}_{\mathcal{U}_n}(X)) \to \mathrm{RipsG}_{\mathcal{U}_m}(X)$ is short for every choice of sufficiently large $m > n$. The maps π_i are a_i-Lipschitz where $\mathcal{U}_i$ is a_i-bounded cover. Also π_n is *ls*-equivalent to $\pi_{n+1} \circ i_{n,n+1}$ for each $n \geqslant 1$.

Let $g : V \to X$ be a-Lipschitz map from a graph V. Then g induces a short map $g' : V \to \mathrm{RipsG}_{\mathcal{U}_n}(X)$ for every n such that $L(\mathcal{U}_n) \geqslant a$ and $\pi_n \circ g' = g$ holds.

Suppose $g, h : V \to \mathrm{RipsG}_{\mathcal{U}_n}(X)$ are two short maps from a graph such that $\pi_n \circ g$ is d-close to $\pi_n \circ h$. Then $i_{n,m} \circ g$ is 1-close to $i_{n,m} \circ h$ for every $m \geqslant n$ so that $L(\mathcal{U}_m) \geqslant d$. $\square$

Note that $i_{n,m} : A(\mathrm{RipsG}_{\mathcal{U}_n}(X)) \to \mathrm{RipsG}_{\mathcal{U}_m}(X)$ is short if $\mathcal{U}_n$ is a star refinement of $\mathcal{U}_m$.

Corollary 3.7. *If $t(n) \to \infty$ is increasing, then the sequence $\{\mathrm{RipsG}_{t(1)}(X) \to \mathrm{RipsG}_{t(2)}(X) \to \cdots\}$ together with projections $\pi_{t(n)} : \mathrm{RipsG}_{t(n)}(X) \to X$ forms a coarse graph of (X, d_X).*

For any two graphs G_1 and G_2 let $\mathrm{Short}(G_1, G_2)_{ls}$ be the set of large scale equivalence classes of short maps from G_1 to G_2.

Given a coarse graph $\{V_1 \to V_2 \to \cdots\}$ and a graph V we consider the direct limit of $\mathrm{Short}(V, V_1)_{ls} \to \mathrm{Short}(V, V_2)_{ls} \to \cdots$ and define it as the set of morphisms from V to $\{V_1 \to V_2 \to \cdots\}$.

The set of morphisms from $\{W_1 \to W_2 \to \cdots\}$ to $\mathcal{V} = \{V_1 \to V_2 \to \cdots\}$ is the inverse limit of $\cdots \to \mathrm{Mor}(W_n, \mathcal{V}) \to \cdots \to \mathrm{Mor}(W_1, \mathcal{V})$.

We can restate the above definition of morphisms between coarse graphs as follows. Suppose $\mathcal{V} = \{V_1 \overset{i_{1,2}}{\to} V_2 \overset{i_{2,3}}{\to} \cdots\}$ and $\mathcal{W} = \{W_1 \overset{j_{1,2}}{\to} W_2 \overset{j_{2,3}}{\to} \cdots\}$ are two coarse graphs. First we consider a *pre-morphism* $F : \mathcal{V} \to \mathcal{W}$, that comes with a function $n_F : \mathbb{N} \to \mathbb{N}$, and consists of short maps $f_k : V_k \to W_{n_F(k)}$ so that for every $k \geqslant 1$ there is $m \geqslant n_F(k+1)$ resulting in $j_{n_F(k),m} \circ f_k \sim_{ls} j_{n_F(k+1),m} \circ f_{k+1} \circ i_{k,k+1}$.

$$\begin{tikzcd}
W_1 \arrow[r, "j_{1,2}"] & W_2 \arrow[r, "j_{2,3}"] & \cdots \arrow[r, "j"] & W_{n_F(1)} \arrow[r, "j"] & \cdots \arrow[r, "j"] & W_{n_F(2)} \arrow[r, "j"] & \cdots \arrow[r, "j"] & W_{n_F(3)} \cdots \\
V_1 \arrow[r, "i_{1,2}"'] \arrow[urrr, "f_1"] & V_2 \arrow[r, "i_{2,3}"'] \arrow[urrrr, "f_2"] & V_3 \arrow[r, "i_{3,4}"'] \arrow[urrrrr, "f_3"'] & \cdots
\end{tikzcd}$$

Two pre-morphisms $F, G : \mathcal{V} \to \mathcal{W}$ are considered to be equivalent if for every k there is $m \geqslant \max\{n_F(k), n_G(k)\}$ so that $j_{n_F(k),m} \circ f_k \sim_{ls} j_{n_G(k),m} \circ g_k$. The sets of equivalence classes of pre-morphisms form the set of morphisms from $\mathcal{V}$ to $\mathcal{W}$.

Definition 3.8. Coarse graphs $\mathcal{V}$ and $\mathcal{W}$ are *ls-equivalent* if there exist pre-morphisms $F : \mathcal{V} \to \mathcal{W}$ and $G : \mathcal{W} \to \mathcal{V}$ such that $G \circ F$ is equivalent to $id_{\mathcal{V}}$ and $F \circ G$ is equivalent to $id_{\mathcal{W}}$.

Theorem 3.9. *If $\mathcal{V} = \{V_1 \to V_2 \to \cdots\}$ is a coarse graph of (X, d_X) and $\mathcal{W} = \{W_1 \to W_2 \to \cdots\}$ is a coarse graph of (Y, d_Y), then there is a natural bijection between bornologous maps from X to Y and morphisms from $\mathcal{V}$ to $\mathcal{W}$.*

Proof. We will follow the notation from the diagram below

$$\begin{tikzcd}
V_1 \arrow[r, "i_{1,2}"] \arrow[d, "p_1"'] & V_2 \arrow[r, "i_{2,3}"] \arrow[dl, "p_2"'] & V_3 \arrow[r, "i_{3,4}"] \arrow[dll, "p_3"] & \cdots & & W_1 \arrow[r, "j_{1,2}"] \arrow[d, "q_1"'] & W_2 \arrow[r, "j_{2,3}"] \arrow[dl, "q_2"'] & W_3 \arrow[r, "j_{3,4}"] \arrow[dll, "q_3"] & \cdots \\
X & & & & & Y & & &
\end{tikzcd}$$

Suppose $f : X \to Y$ is a bornologous map. Notice that every map $f \circ p_i : V_i \to Y$ is Lipschitz as f is bornologous and $f \circ p_i$ is defined on a graph. As $\mathcal{W}$ is a coarse graph of (Y, d_Y), each $f \circ p_i$ lifts (up to $\sim_{ls}$) to a short map $f_i : V_i \to W_{n(i)}$. Since $\mathcal{V}$ is a coarse graph of (X, d_X), maps f_i form a pre-morphism $F : \mathcal{V} \to \mathcal{W}$. Also note that such construction gives us a unique morphism F with the property $f \circ p_* \sim_{ls} q_* \circ F$.

Let $F : \mathcal{V} \to \mathcal{W}$ be a pre-morphism, let $\{\mathrm{RipsG}_1(X) \overset{e_{1,2}}{\to} \mathrm{RipsG}_2(X) \overset{e_{2,3}}{\to} \cdots\}$ be the Rips coarse graph of (X, d_X) and let $\pi_n : \mathrm{RipsG}_n(X) \to X$ be the projections. The identity map $id_X \circ \pi_k : \mathrm{RipsG}_k(X) \to X$ is k-Lipschitz and lifts (up to $\sim_{ls}$) to a short map $g_k : \mathrm{RipsG}_k(X) \to V_{m(k)}$. Define a map $f : X \to Y$ by $f(x) := q_{n(m(1))} \circ f_{m(1)} \circ g_1(x)$. Note that the properties of coarse graphs and their morphisms imply that all the maps $X \to Y$ defined by $f(x) := q_{n(m(k))} \circ f_{m(k)} \circ g_k(x)$ are ls-equivalent to each other which means there is exactly one function f factoring over F. Also note that maps $q_{n(m(k))} \circ f_{m(k)} \circ g_k : \mathrm{RipsG}_k(X) \to Y$ are Lipschitz for every choice of k which means that f is bornologous: if $d_X(x, y) < r$ and $k \in \mathbb{N}$ is such that $k \geqslant r$, then $d_Y(f(x), f(y)) < L + 2C$ where L is the Lipschitz constant of the map $q_{n(m(k))} \circ f_{m(k)} \circ g_k$ and $q_{n(m(k))} \circ f_{m(k)} \circ g_k$ is C-close to $q_{n(m(1))} \circ f_{m(1)} \circ g_1$

$$
\begin{tikzcd}
 & X \arrow[rr, "id_X"] & & X \arrow[rr, dashed, "f"] & & Y \\
\mathrm{RipsG}_1(X) \arrow[ur, "\pi"] \arrow[d, "e"] \arrow[drr, Rightarrow, dashed, "g_1"] & & V_1 \arrow[ur, "\mathbf{p}"] \arrow[d, "i"] & & W_1 \arrow[ur, "\mathbf{q}"] \arrow[d, "j"] & \\
\mathrm{RipsG}_2(X) \arrow[uur] \arrow[d, "e"] \arrow[ddrr, Rightarrow, dashed, "g_2"] & & V_2 \arrow[uur] \arrow[d, "i"] \arrow[ddrr, Rightarrow, "f_2"] & & W_2 \arrow[uur] \arrow[d, "j"] & \\
\mathrm{RipsG}_3(X) \arrow[uuur] \arrow[d, "e"] & & V_3 \arrow[uuur] \arrow[d, "i"] & & W_3 \arrow[uuur] \arrow[d, "j"] & \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

The fact that f is a unique map that factors (up to $\sim_{ls}$) over F (that is $f \circ p_* \sim_{ls} q_* \circ F$) implies that the rule $F \mapsto f$ is the inverse of the rule $f \mapsto F$ from the previous paragraph. Hence the two rules induce a bijection.

These bijections are natural as $f \circ g \mapsto F \circ G$ which follows from the *ls*-commutativity of the diagrams. □

Corollary 3.10. *Any two coarse graphs of X are ls-equivalent.*

4. Coarse simplicial complexes

We have seen that graphs are sufficient to describe coarse category of metric spaces. However, in order to capture more complicated concepts (asymptotic dimension, coarse connectivity, and Property A) we need to consider simplicial complexes.

In this section we will consider simplicial complexes K with the set of vertices X. Each such complex induces the graph $G(K)$ obtained by considering only the edges of K (i.e., $G(K)$ is the 1-skeleton of K).

Proposition 4.1. *A function $f : G(K) \to Y$ is Lipschitz if and only if the family $\{f(\Delta)\}_{\Delta \in K}$ is uniformly bounded in (Y, d_Y).*

Proof. If f is a-Lipschitz then $\{f(\Delta)\}_{\Delta \in K}$ is uniformly a-bounded. Conversely, if $\{f(\Delta)\}_{\Delta \in K}$ is uniformly a-bounded then f is a-Lipschitz as graphs are 1-geodesic. □

Conversely, each graph G on X induces the minimal flag complex $F(G)$ containing G (recall K is a *flag complex* if Δ is a simplex of K whenever $[v, w]$ belongs to K for all $v, w \in \Delta$) and each short map $G_1 \to G_2$ induces a simplicial map $F(G_1) \to F(G_2)$.

Example 4.2. Given a metric space (X, d) and a uniformly bounded cover $\mathcal{U}$ of X, the *Rips complex* $\mathrm{Rips}_{\mathcal{U}}(X)$ equals $F(\mathrm{RipsG}_{\mathcal{U}}(X))$. If $\mathcal{U}_r = \{B(x, r)\}_{x \in X}$ is the cover of X by all closed balls of radius r, then the nerve $\mathcal{N}(\mathcal{U}_r)$ and $\mathrm{Rips}_{\mathcal{U}_r}(X)$ have identical 1-skeleton for each positive r. Hence the natural simplicial inclusion $\mathcal{N}(\mathcal{U}_r) \to \mathrm{Rips}_{\mathcal{U}_r}(X)$ is an *ls*-equivalence.

We can extend the definition of $A(G)$ from graphs to complexes as follows:

Definition 4.3. $\Delta \in A(K)$ if and only if there is $v \in K^{(0)}$ such that the set of vertices $\Delta^{(0)}$ of Δ is contained in the closed star of v in K. Equivalently, there is a vertex v of K such for each vertex w of Δ we have $w = v$ or $[v, w]$ is an edge of K.

A map $f : K \to X$ from a simplicial complex K to a metric space X is *bornologous* if and only if the family $\{f(\Delta)\}_{\Delta \in K}$ is uniformly bounded.

Note that any coarse graph $G_1 \to G_2 \to \cdots$ induces simplicial maps of complexes $F(G_n) \to F(G_m)$.

Definition 4.4. A *coarse simplicial complex* is a direct sequence $\{K_1 \to K_2 \to \cdots\}$ of simplicial complexes K_n and simplicial maps $i_{n,m} : K_n \to K_m$ for all $n \leqslant m$ such that the following conditions are satisfied:

a. $i_{n,n} = id$ and $i_{n,k} = i_{m,k} \circ i_{n,m}$ for all $n \leqslant m \leqslant k$,
b. for each n there is $m > n$ such that $i_{n,m} : A(K_n) \to K_m$ is simplicial.

Example 4.5. Recall that for a metric space (X, d_X) and $t > 0$, the Rips complex $\mathrm{Rips}_t(X)$ consists of simplices $[x_0, x_1, \ldots, x_n]$ such that $\mathrm{diam}\{x_0, x_1, \ldots, x_n\} \leqslant t$ and $x_0, x_1, \ldots, x_n \in X$. For any increasing sequence $\{t(n)\}$ diverging to infinity, the sequence of Rips complexes $K_n = \mathrm{Rips}_{t(n)}(X)$ with identity maps $i_{n,m} : K_n \to K_m$ for $n \leqslant m$ forms a coarse simplicial complex.

Example 4.6. Suppose (X, d_X) is a metric space and $\mathcal{U}_n$ is a sequence of uniformly bounded covers of X such that Lebesgue numbers $L(\mathcal{U}_n) \to \infty$ as $n \to \infty$ and $\mathcal{U}_n$ is a refinement of $\mathcal{U}_{n+1}$. Then the sequence of Rips complexes $K_{\mathcal{U}_n} = \mathrm{Rips}_{\mathcal{U}_n}(X)$ with identity maps $i_{n,m} : K_n \to K_m$ for $n \leqslant m$ forms a coarse simplicial complex. Any such coarse complex will be denoted by $\mathrm{Rips}_*(X)$ and called a *coarse Rips complex of* X.

Proof. Similar as first part of Proposition 3.6. Because $\mathcal{U}_n$ is a refinement of $\mathcal{U}_{n+1}$, $[v, w]$ is an edge of $\mathrm{Rips}_{\mathcal{U}_m}(X)$ for every edge $[v, w]$ of $\mathrm{Rips}_{\mathcal{U}_n}(X)$ with $n \leqslant m$. $\mathrm{Rips}_*(X)$ consisting of flag complexes of their one skeleta implies that all the maps $i_{n,m}$ are simplicial. Furthermore, edges of $A(\mathrm{Rips}_{\mathcal{U}_n}(X))$ form a subset of edges of $\mathrm{Rips}_{\mathcal{U}_m}(X)$ for every m with $L(\mathcal{U}_m) \geqslant d$ (where $\mathcal{U}_n$ is d bounded), hence natural map $A(\mathrm{Rips}_{\mathcal{U}_n}(X)) \to \mathrm{Rips}_{\mathcal{U}_m}(X)$ is simplicial. $\square$

The following is similar to Roe's concept of an anti-Čech approximation of a metric space X:

Example 4.7. Suppose (X, d_X) is a metric space and $\mathcal{U}_n$ is a sequence of uniformly bounded covers of X such that $\mathcal{U}_{n+1}$ is a star refinement of $\mathcal{U}_n$ for each $n \geqslant 1$. Then the sequence $\mathcal{N}(\mathcal{U}_1) \to \mathcal{N}(\mathcal{U}_2) \to \cdots$ of nerves of covers $\mathcal{U}_n$ forms a coarse simplicial complex if $i_{n,n+1}(U)$ contains the star $st(U, \mathcal{U}_n)$ for each $U \in \mathcal{U}_n$. Any such coarse complex will be denoted by $\check{\mathrm{C}}\mathrm{ech}_*(X)$ and called a *coarse Čech complex of* X.

Proof. For $U \in \mathcal{N}(\mathcal{U}_n)$ define $i_{n,n+1}(U)$ to be any $V \in \mathcal{N}(\mathcal{U}_{n+1})$ that contains the star of U. Maps $i_{n,n+1}(U)$ induce maps $i_{n,m}$. To prove that $i_{n,n+1}$ is simplicial consider simplex $[U_1, \ldots, U_k] \in \mathcal{N}(\mathcal{U}_n)$ i.e. $U_1 \cap \cdots \cap U_k \neq \emptyset$. This, for any $f : X \to X$, implies $f(U_1) \cap \cdots \cap f(U_k) \neq \emptyset$, i.e., $[f(U_1), \ldots, f(U_k)] \in \mathcal{N}(\mathcal{U}_{n+1})$ hence map $i_{n,n+1}$ is simplicial $\forall n$.

Suppose each of the sets $U_1, \ldots, U_k \in \mathcal{U}_n$ has nonempty intersection with $U \in \mathcal{U}_n$. Then all the sets $i_{n,n+1}(U_1), \ldots, i_{n,n+1}(U_k)$ have nonempty intersection (namely contain the set U) hence $[i_{n,n+1}(U_1), \ldots, i_{n,n+1}(U_k)] \in \mathcal{N}(\mathcal{U}_{n+1})$ which implies that $i_{n,n+1} : A(\mathcal{N}(\mathcal{U}_n)) \to \mathcal{N}(\mathcal{U}_{n+1})$ is simplicial. $\square$

Definition 4.8. Simplicial maps $f, g : K \to L$ between simplicial complexes are *contiguous* if for every simplex Δ of K, $f(\Delta) \cup g(\Delta)$ is contained in some simplex of L.

Note that contiguity is not an equivalence relation.

Lemma 4.9. *Let $\mathcal{K} = \{K_1 \xrightarrow{i_{1,2}} K_2 \xrightarrow{i_{2,3}} \cdots\}$ be a coarse simplicial complex and suppose $\alpha, \beta, \gamma : L \to K_n$ are maps defined on a simplicial complex L for some n. If α is contiguous to β and β is contiguous to γ, then there exists $m > n$ such that $i_{n,m}\alpha$ is contiguous to $i_{n,m}\gamma$.*

Proof. By the definition of the coarse simplicial complex there exists $m > n$ such that $i_{n,m} : A(K_n) \to K_m$ is simplicial. For every simplex Δ of L, $\alpha(\Delta) \cup \beta(\Delta) \subset \rho_1$ and $\beta(\Delta) \cup \gamma(\Delta) \subset \rho_2$ for some simplices ρ_1 and ρ_2 of K_n. Since $\rho_1 \cap \rho_2 \neq \emptyset$ the union $\rho_1 \cup \rho_2$ is contained in a star of every vertex of $\rho_1 \cap \rho_2$. Consequently $\rho_1 \cup \rho_2$ is contained in some simplex of $A(K_n)$. Hence $i_{n,m}\alpha(\Delta) \cup i_{n,m}\gamma(\Delta)$ is contained in some simplex of K_m. $\square$

Definition 4.10. A *coarse complex of a metric space* (X, d_X) is a coarse complex $\mathcal{K} = \{K_1 \to K_2 \to \cdots\}$ together with bornologous functions $p_n : K_n \to X$, $n \geqslant 1$, satisfying conditions

a. p_n is *ls*-equivalent to $p_{n+1} \circ i_{n,n+1}$ for each $n \geqslant 1$.
b. For each bornologous function $f : K \to X$ from a simplicial complex K to X there is $n \geqslant 1$ and a simplicial function $g : K \to K_n$ such that f is *ls*-equivalent to $p_n \circ g$.
c. If $f, g : K \to K_n$ are two simplicial functions so that $p_n \circ f \sim_{ls} p_n \circ g$, then there is $m > n$ such that $i_{n,m} \circ f$ is contiguous to $i_{n,m} \circ g$.

Example 4.11. Any coarse Rips complex $\mathrm{Rips}_*(X)$ together with identity functions $\mathrm{Rips}_*(X) \to X$ forms a coarse complex of X, if the Lebesgue numbers $L(\mathcal{U}_n) \to \infty$ as $n \to \infty$.

Proof. Suppose $\Delta := [v_1, \ldots, v_k] \in \mathrm{Rips}_{\mathcal{U}_n}(X)$ where $\mathcal{U}_n$ is d_n-bounded cover of X. Then $p_n(\Delta)$ is d_n-bounded hence p_n is bornologous. Also note that $p_n = p_{n+1} \circ i_{n,n+1}$ by the definition.

Suppose $f : K \to X$ is a bornologous function from a simplicial complex K to X so that sets $\{f(\Delta)\}_{\Delta \in K}$ are a-bounded. Then the naturally induced map $f_n : K \to \mathrm{Rips}_{\mathcal{U}_n}(X)$ (naturally induced meaning $p_n \circ f_n = f$) is simplicial for every n with $L(\mathcal{U}_n) \geqslant a$.

Suppose $f, g : K \to K_n$ are two simplicial functions so that $p_n \circ f$ is d-close to $p_n \circ g$. Then $i_{n,m} \circ f$ is contiguous to $i_{n,m} \circ g$ for every $m \geqslant n$ with $L(\mathcal{U}_m) \geqslant d + b$, where $\mathcal{U}_n$ is b-bounded. $\square$

Example 4.12. Any coarse Čech complex $\check{\mathrm{C}}\mathrm{ech}_*(X)$ with maps $p_n : \mathcal{N}(\mathcal{U}_n) \to X$ such that $p_n(U) \in U$ for all $U \in \mathcal{U}_n$ forms a coarse complex of X.

Proof. Define $p_n(U)$ to be any point of U. If the cover $\mathcal{U}_n$ is a_n-bounded then p_n is $2a_n$-bornologous and p_n is a_{n+1}-close to $p_{n+1} \circ i_{n,n+1}$.

Suppose $f : K \to X$ is a b-bornologous map from a simplicial complex K to X. Pick n so that $L(\mathcal{U}_n) \geqslant b$ and define $g : K \to \mathcal{N}(\mathcal{U}_n)$ by mapping v to any element of $\mathcal{U}_n$ containing $B(f(v), b)$. Note that $p_n \circ g$ is a_n-close to f. Furthermore g is simplicial: if $[v_1, \ldots, v_k] \in K$ then $f(v_i) \in g(v_j), \forall i, j$, hence $[g(v_1), \ldots, g(v_k)] \in \mathcal{N}(\mathcal{U}_n)$.

Suppose $g, f : K \to \mathcal{N}(\mathcal{U}_n)$ are two simplicial maps, so that $p_n \circ f$ and $p_n \circ g$ are d-close and b-bornologous (if p_n is b-bornologous and f is simplicial, then $p_n \circ f$ is b-bornologous). Choose $m \geqslant n$ so that $L(\mathcal{U}_m) \geqslant d + b$ and let $\Delta = [v_1, \ldots, v_k] \in K$. There exists $U \in \mathcal{U}_m$ containing $B(p_n \circ f(v_1), d + b)$ hence it contains $p_n \circ f(v_i), p_n \circ g(v_j), \forall i, j$. Such set will be contained in all sets $i_{n,m+1} \circ f(v_i)$ and $i_{n,m+1} \circ g(v_j)$ hence $i_{n,m+1} \circ f(\Delta) \cup i_{n,m+1} \circ g(\Delta)$ is contained in a simplex of $\mathcal{N}(\mathcal{U}_{m+1})$. The parameter m does not depend on the choice of Δ which implies that $i_{n,m+1} \circ f$ and $i_{n,m+1} \circ g$ are contiguous. $\square$

We define a morphism between coarse simplicial complexes in the same we as we did for morphism between coarse graphs. Suppose $\mathcal{L} = \{L_1 \overset{i_{1,2}}{\to} L_2 \overset{i_{2,3}}{\to} \cdots\}$ and $\mathcal{K} = \{K_1 \overset{j_{1,2}}{\to} K_2 \overset{j_{2,3}}{\to} \cdots\}$ are two coarse simplicial complexes. First we consider a *pre-morphism* $F : \mathcal{L} \to \mathcal{K}$, that comes with a function $n_F : \mathbb{N} \to \mathbb{N}$, and consists of simplicial maps $f_k : L_k \to K_{n_F(k)}$ so that for every $k \geqslant 1$ there is $m \geqslant n_F(k+1)$ such that $j_{n_F(k),m} \circ f_k$ and $j_{n_F(k+1),m} \circ f_{k+1} \circ i_{k,k+1}$ are contiguous.

$$\begin{tikzcd}
K_1 \arrow[r, "j_{1,2}"] & K_2 \arrow[r, "j_{2,3}"] & \cdots \arrow[r, "j"] & K_{n_F(1)} \arrow[r, "j"] & \cdots \arrow[r, "j"] & K_{n_F(2)} \arrow[r, "j"] & \cdots \arrow[r, "j"] & K_{n_F(3)} \cdots \\
L_1 \arrow[r, "i_{1,2}"'] \arrow[urrr, "f_1"] & L_2 \arrow[r, "i_{2,3}"'] \arrow[urrrr, "f_2"] & L_3 \arrow[r, "i_{3,4}"'] \arrow[urrrrr, "f_3"] & \cdots
\end{tikzcd}$$

Two pre-morphisms $F, G : \mathcal{L} \to \mathcal{K}$ are considered to be equivalent if for every k there is $m \geqslant \max\{n_F(k), n_G(k)\}$ so that $j_{n_F(k),m} \circ f_k$ and $j_{n_G(k),m} \circ g_k$ are contiguous. The sets of equivalence classes of pre-morphisms form the set of morphisms from $\mathcal{L}$ to $\mathcal{K}$.

Definition 4.13. Coarse simplicial complexes $\mathcal{L}$ and $\mathcal{K}$ are *ls-equivalent* if there exist pre-morphisms $F : \mathcal{L} \to \mathcal{K}$ and $G : \mathcal{K} \to \mathcal{L}$ such that $G \circ F$ is equivalent to $id_{\mathcal{L}}$ and $F \circ G$ is equivalent to $id_{\mathcal{K}}$.

Proposition 4.14. *If (X, d_X) and (Y, d_Y) are metric spaces with coarse complexes $\mathcal{K}$ and $\mathcal{L}$ respectively, then there is a natural bijection between the set of bornologous maps from X to Y and the set of morphisms from $\mathcal{K}$ to $\mathcal{L}$.*

Proof. The proof amounts to a modification of the proof of Theorem 3.9. $\square$

Corollary 4.15. *Any coarse Rips complex of X and any coarse Čech complex of X are ls-equivalent.*

5. Asymptotic dimension of coarse simplicial complexes

Definition 5.1. We say that $K \overset{g}{\to} M \overset{h}{\to} L$ is a *contiguous factorization* of $K \overset{f}{\to} L$ if f, g, and h are simplicial maps and f is contiguous to $h \circ g$.

Definition 5.2. Given a coarse simplicial complex $\mathcal{K}$ we say its *asymptotic dimension* is at most n (notation: $\operatorname{asdim}(\mathcal{K}) \leqslant n$) if for each m there is $k > m$ such that $i_{m,k}$ factors contiguously through an n-dimensional simplicial complex.

Corollary 5.3. *If $\mathcal{K}$ is ls-equivalent to $\mathcal{L}$, then* $\operatorname{asdim}(\mathcal{K}) = \operatorname{asdim}(\mathcal{L})$.

Proof. Let $\mathcal{K} = \{K_1 \overset{i_{1,2}}{\to} K_2 \overset{i_{2,3}}{\to} \cdots\}$ and $\mathcal{L} = \{L_1 \overset{j_{1,2}}{\to} L_2 \overset{j_{2,3}}{\to} \cdots\}$ be large scale equivalent. Let $\varphi : \mathcal{K} \to \mathcal{L}$ be an isomorphism and $\psi : \mathcal{L} \to \mathcal{K}$ its inverse. Suppose $\operatorname{asdim}(\mathcal{K}) \leqslant n$. For every m there exist $k > n_\psi(m)$, an n-dimensional simplicial complex M and simplicial maps $f : K_{n_\psi(m)} \to M$ and $g : M \to K_k$ such that gf and $i_{n_\psi(m),k}$ are contiguous. Because $\varphi\psi$ is equivalent to $id_{\mathcal{L}}$ there exists $l > n_\varphi(k)$ such that $j_{m,l}$ and $j_{n_\varphi(k),l}\varphi_k i_{n_\psi(m),k}\psi_m$ are contiguous. Since $j_{n_\varphi(k),l}\varphi_k i_{n_\psi(m),k}\psi_m$ is contiguous to $j_{n_\varphi(k),l}\varphi_k g f \psi_m$, Lemma 4.9 provides an index $t > l$ such that $j_{m,t}$ is contiguous to $j_{n_\varphi(k),t}\varphi_k g f \psi_m$. Let $\tilde{f} = f\psi_m$ and $\tilde{g} = j_{n_\varphi(k),t}\varphi_k g$, then $j_{m,t}$ is contiguous to $\tilde{g}\tilde{f}$, hence $\operatorname{asdim}(\mathcal{L}) \leqslant n$. $\square$

Recall that asymptotic dimension of metric space X is at most n (notation: $\operatorname{asdim}(X) \leqslant n$) if for each uniformly bounded cover $\mathcal{U}$ of X there is a uniformly bounded cover $\mathcal{V}$ of X of multiplicity $\leqslant n+1$ so that $\mathcal{U}$ refines $\mathcal{V}$. The statement $\operatorname{asdim}(X) \leqslant n$ is equivalent to the following: for every $t > 0$ there exists a uniformly bounded cover $\mathcal{W}$ of X of multiplicity $\leqslant n+1$ and with Lebesgue number $\geqslant t$.

Theorem 5.4. $\operatorname{asdim}(X) = \operatorname{asdim}(\mathrm{Rips}_*(X))$ *for any metric space* X.

Proof. Let $\operatorname{asdim}(\mathrm{Rips}_*(X)) \leqslant n$. By Corollaries 5.3 and 4.15 the same bound holds for the asymptotic dimension of any Čech coarse complex of X. For the rest of the proof we will consider Čech coarse complexes instead of Rips coarse complexes. Given $t > 0$ consider the cover $\mathcal{U} = \{B(x,t)\}_{x\in X}$ of X by t-balls (corresponding to $\mathcal{U}_m$ in $\mathcal{N}(\mathcal{U}_m) = K_m$ of a Čech coarse complex in Definition 5.2), and choose a uniformly bounded cover $\mathcal{V} = \{V_i\}_{i\in J}$ of X together with a function $f : X \to J$ such that $B(x,t) \subset V_{f(x)}$ for all $x \in X$ (this $\mathcal{V}$ corresponds to $\mathcal{U}_k$ in Definition 5.2). Now f factors contiguously through an n-dimensional simplicial complex K as $X \xrightarrow{g} L \xrightarrow{h} J$, where L is the set of vertices of K.

Given $l \in L$ define W_l as the union of all $B(x,t)$ such that $g(x) = l$. Let us show the multiplicity of $\mathcal{W} = \{W_l\}_{l\in L}$ is at most $n+1$. Suppose, on the contrary, that there are mutually different elements $l(0),\dots,l(n+1)$ of L such that $W_{l(0)} \cap \dots \cap W_{l(n+1)} \neq \emptyset$. That means existence of $x \in X$ and elements $x(0),\dots,x(n+1)$ of X such that $x \in B(x(k),t)$ and $g(x(k)) = l(k)$ for all $0 \leqslant k \leqslant n+1$. Since g is a simplicial map and $[x(0),\dots,x(n+1)]$ is a simplex in the nerve $\mathcal{N}(\mathcal{U})$ of $\mathcal{U}$, $[l(0),\dots,l(n+1)]$ is an $(n+1)$-dimensional simplex in K contradicting K being n-dimensional.

It remains to show $\mathcal{W}$ is uniformly bounded as its Lebesgue number is at least t. Given $l \in L$ put $j = h(l)$. If $g(x) = l$ and $h \circ g$ is contiguous to f, $[f(x), j]$ is a simplex in $\mathcal{N}(\mathcal{V})$, so $V_{f(x)} \cap V_j \neq \emptyset$. That means W_l is contained in the star of V_j in $\mathcal{V}$. Therefore $\mathcal{W}$ is uniformly bounded.

If $\operatorname{asdim}(X) \leqslant n$ then there exists a Čech coarse complex of X which consists of n-dimensional simplicial complexes. □

6. Connectivity of coarse simplicial complexes

Here is the basic extension of connectedness to large scale geometry of metric spaces:

Definition 6.1. ([1, Definition 2.10]) A metric space X is *coarsely k-connected* if for each r there exists $R \geqslant r$ so that the mapping $|\mathrm{Rips}_r(X)| \to |\mathrm{Rips}_R(X)|$ induces a trivial map of π_i for $0 \leqslant i \leqslant k$.

It has a natural generalization to coarse simplicial complexes:

Definition 6.2. A coarse simplicial complex $\mathcal{K}$ is *coarsely n-connected* if for each m there is $k > m$ such that $i_{m,k}$ induces trivial homomorphisms $\pi_p(i_{m,k})$ of homotopy groups for all $0 \leqslant p \leqslant n$.

Thus X is coarsely n-connected if and only if $\mathrm{Rips}_*(X)$ is coarsely n-connected.

Corollary 2.15 of [1] says that coarse k-connectedness is a quasi-isometry invariant. We can easily generalize it slightly:

Corollary 6.3. *Coarse k-connectedness is a large scale invariant.*

Recall that a metric space (X, d_X) is *t-chain connected* for some $t > 0$ if for every two points x, y of X there is a t-chain joining them (that means every two consecutive points in the chain are at distance at most t). Alternatively, $\mathrm{Rips}_t(X)$ is connected. Let us show X being coarsely 0-connected and X being t-chain connected for some $t > 0$ are equivalent concepts.

Proposition 6.4. *If* (X, d_X) *is a metric space, then the following conditions are equivalent*:

a. *X is coarsely* 0*-connected,*
b. *there is* $t > 0$ *such that* $\mathrm{Rips}_t(X)$ *is connected,*
c. *there is* $t > 0$ *such that* $\mathrm{Rips}_s(X)$ *is connected for all* $s \geqslant t$*,*
d. d_X *attains only finite values and there is* $t > 0$ *such that* $H_0(\mathrm{Rips}_t(X)) \to H_0(\mathrm{Rips}_s(X))$ *is injective for all* $s \geqslant t$.

Proof. (a) ⇒ (b). Choose $t > 1$ such that the image of $\mathrm{Rips}_1(X)$ in $\mathrm{Rips}_t(X)$ is contained in one path-component of $\mathrm{Rips}_t(X)$. Given two points $x, y \in X$, the corresponding vertices x and y in $\mathrm{Rips}_t(X)$ have to be joinable by a sequence of edges, i.e. x and y are joinable in X by a t-chain.

(b) ⇒ (c). If every two points of X are joinable by a t-chain, they are joinable by s-chain for any $s \geqslant t$ (use the same chain).

(c) ⇒ (d) is obvious.

(d) ⇒ (a). Notice the direct limit of $\tilde{H}_0(\mathrm{Rips}_n(X)) \to \tilde{H}_0(\mathrm{Rips}_{n+1}(X)) \to \cdots$ is trivial and $\tilde{H}_0(\mathrm{Rips}_t(X))$ maps to it in an injective manner if $n > t$. Therefore $\tilde{H}_0(\mathrm{Rips}_t(X)) = 0$ which means X is t-chain connected. □

Definition 6.5. A metric space (X, d_X) is *coarsely geodesic* if it is coarsely equivalent to a geodesic metric space.

Proposition 6.6. *Let (X, d_X) be a metric space. The following conditions are equivalent*:

a. *(X, d_X) is coarsely geodesic,*
b. *(X, d_X) coarsely 0-connected and there is $t > 0$ such that the identity function $(X, d_X) \to \mathrm{RipsG}_t(X)$ is bornologous,*
c. *(X, d_X) coarsely 0-connected and there is $t > 0$ such that the identity function $(X, d_X) \to \mathrm{RipsG}_t(X)$ is a coarse equivalence,*
d. *(X, d_X) coarsely 0-connected and there is $t > 0$ such that the identity function $(X, d_X) \to \mathrm{RipsG}_s(X)$ is a coarse equivalence for all $s \geqslant t$.*

Proof. Clearly, (X, d_X) coarsely geodesic $\Rightarrow$ (X, d_X) coarsely 0-connected.

If (Y, d_Y) is geodesic, then the identity function $(Y, d_Y) \to \mathrm{RipsG}_t(Y)$ is large scale Lipschitz for all $t > 0$, so the identity function $(X, d_X) \to \mathrm{RipsG}_t(X)$ is bornologous for every coarsely geodesic space (X, d_X) and $t > 0$ sufficiently large.

Suppose the identity function $(X, d_X) \to \mathrm{RipsG}_t(X)$ is bornologous for some $t > 0$. Since the identity function $\mathrm{RipsG}_t(X) \to (X, d_X)$ is t-Lipschitz, $\mathrm{RipsG}_t(X)$ and (X, d_X) are coarsely equivalent. Since $\mathrm{RipsG}_t(X)$ is coarsely equivalent to its geometric realization (which is geodesic if X is coarsely 0-connected and t is sufficiently large), we are done. □

Corollary 6.7. *Every t-geodesic metric space (X, d_X) is coarsely geodesic.*

Proof. Observe that identity function $X \to \mathrm{RipsG}_t(X)$ is bornologous. Indeed, given $x, y \in X$ choose a t-chain $x_0, \ldots, x_k$ joining x and y such that $d_X(x, y) = \sum_{i=0}^{k-1} d_X(x_i, x_{i+1})$ and k is minimal with respect to that property. Therefore either $d_X(x, y) \leqslant 2 \cdot t$ or $d_X(x, y) > 2 \cdot t$, $k \geqslant 3$, and $d_X(x_i, x_{i+1}) + d_X(x_{i+1}, x_{i+2}) > t$ for each $0 \leqslant i \leqslant k-2$. That implies $2 \cdot d_X(x, y) \geqslant (k-1) \cdot t$. Since k is greater than or equal to the distance of x and y in $\mathrm{RipsG}_t(X)$, $X \to \mathrm{RipsG}_t(X)$ is bornologous. □

Definition 6.8. A coarse simplicial complex $\mathcal{K}$ is *coarsely homology n-connected* if for each m there is $k > m$ such that $i_{m,k}$ induces trivial homomorphisms $\tilde{H}_p(i_{m,k})$ of reduced homology groups for all $0 \leqslant p \leqslant n$.

A metric space X is *coarsely homology n-connected* if for each r there exists $R \geqslant r$ so that the mapping $|\mathrm{Rips}_r(X)| \to |\mathrm{Rips}_R(X)|$ induces a trivial map of reduced homology groups $\tilde{H}_i$ for $0 \leqslant i \leqslant n$.

$H_1(X)$ being *uniformly generated* (see [4]) means there is $R > 0$ such that every loop in X is homologous to the sum of loops, each of diameter at most R. $\pi_1(X)$ being *uniformly generated* (see [4]) means there is $R > 0$ such that every loop in X extends over a perforated disk D and the image of each internal hole of D has diameter at most R.

Fujiwara and Whyte [4] proved that every complete geodesic space X so that $H_1(X)$ is uniformly generated (respectively, $\pi_1(X)$ is uniformly generated) is quasi-isometric to a complete geodesic space Y satisfying $H_1(Y) = 0$ (respectively, $\pi_1(Y) = 0$). Their proof involves adding cones over a family of balls to X. Our next result shows that one can use Rips complexes for the same purpose.

Proposition 6.9. *If X is a geodesic metric space, then the following conditions are equivalent*:

a. *X is coarsely homology 1-connected (respectively, coarsely 1-connected),*
b. *there is $t > 0$ such that $H_1(\mathrm{Rips}_t(X)) = 0$ (respectively, $\mathrm{Rips}_t(X)$ is simply connected),*
c. *there is $t > 0$ such that $H_1(\mathrm{Rips}_s(X)) = 0$ for all $s \geqslant t$ (respectively, $\mathrm{Rips}_s(X)$ is simply connected for all $s \geqslant t$),*
d. *$H_1(X)$ is uniformly generated (respectively, $\pi_1(X)$ is uniformly generated).*

Proof. If $t < s$, then $H_1(\mathrm{Rips}_t(X)) \to H_1(\mathrm{Rips}_s(X))$ is an epimorphism as X is geodesic. This observation takes care of implications (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c).

(c) $\Rightarrow$ (d). If $H_1(\mathrm{Rips}_t(X)) = 0$ for some $t > 0$, then any loop can be approximated by a piecewise-geodesic loop, so it suffices to show that any piecewise-geodesic loop γ is homologous to a sum of loops of diameter at most $3t$. Notice γ can be realized in $\mathrm{Rips}_t(X)$, so it is homologous to a sum of loops representing boundaries of 2-simplices in $\mathrm{Rips}_t(X)$, hence their diameter is at most $3t$.

(d) $\Rightarrow$ (c). Consider $t > 0$ such that any element of $H_1(X)$ is homologous to the sum of loops of diameter less than t. Given an element γ of $H_1(\mathrm{Rips}_s(X))$ for any $s \geqslant 2t$ we can realize it as a loop $\gamma : S^1 \to X$ in X then extend over an oriented 2-manifold M, so that its boundary ∂M is the union $S_0 \cup S_1 \cup \cdots \cup S_k$, where $S_0 = S^1$ and $\gamma(S_i)$ is of diameter less than t if $i > 0$. We can triangulate M requiring that $\gamma(\Delta)$ has diameter less than t for each simplex Δ of the triangulation. This triangulation induces a simplicial map from M to $\mathrm{Rips}_s(X)$ such that for every $i > 0$ the image $\gamma(S_i)$ is contained in a simplex of the complex $\mathrm{Rips}_s(X)$. Consequently the map can be extended over cones of each S_i, $i > 0$, thus showing that γ is homologous to 0 in $H_1(\mathrm{Rips}_s(X))$.

A similar proof works in the case of the fundamental groups. □

Corollary 6.10. *If X is a coarsely geodesic metric space, then X is coarsely equivalent to a simply connected geodesic space if and only if X is coarsely 1-connected.*

Proof. By definition X is coarsely equivalent to a geodesic metric space Y.

($\Rightarrow$) By the above proposition (see implication d. $\Rightarrow$ a.), a geodesic 1-connected metric space is also coarsely 1-connected.

($\Leftarrow$) By the above proposition (see implication a. $\Rightarrow$ b.) Y is coarsely equivalent to the 1-connected geodesic space $\mathrm{Rips}_t(Y)$ for some t. $\square$

7. Coarse trees

The purpose of this section is to provide a simple proof of a result of Fujiwara and Whyte [4] (see Corollary 7.2).

Theorem 7.1. *If X is a coarsely geodesic metric space, then the following conditions are equivalent*:

a. *X is coarsely equivalent to a simplicial tree,*
b. $\mathrm{asdim}(X) \leqslant 1$ *and X is coarsely homology* 1*-connected,*
c. $\mathrm{asdim}(X) \leqslant 1$ *and X is coarsely* 1*-connected.*

Proof. Assume (X, d_X) is geodesic.

(b) $\Rightarrow$ (c). There is $s > t$ such that $H_1(\mathrm{Rips}_t(X)) = 0$ and $\mathrm{Rips}_t(X) \to \mathrm{Rips}_s(X)$ contiguously factors through a 1-complex K. Since the image of $\pi_1(\mathrm{Rips}_t(X))$ in $\pi_1(K)$ is both perfect and free, it must be trivial.

(c) $\Rightarrow$ (a). Pick a contiguous factorization $\mathrm{Rips}_t(X) \xrightarrow{f} K \xrightarrow{g} \mathrm{Rips}_s(X)$ such that K is a simplicial tree and both projections $\pi_t : \mathrm{RipsG}_t(X) \to X$ and $\pi_s : \mathrm{RipsG}_s(X) \to X$ are coarse equivalences. Such a factorization exists as we can replace K by its universal cover. We may assume $K = f(\mathrm{RipsG}_t(X))$ as $\mathrm{RipsG}_t(X)$ is connected and we will use the same notation for f considered as a function defined on X and $g : K \to X$. Therefore f and g are bornologous and $g \circ f \sim_{ls} id_X$. As f is bornologous, there is $M > 0$ such that $d_K(f \circ g \circ f(x), f(x)) < M$ for all $x \in X$. Given a vertex v of K pick $x \in X$ so that $v = f(x)$. Now $d_K(f \circ g(v), v) < M$ proving $f \circ g \sim_{ls} id_K$. $\square$

Corollary 7.2. *(Fujiwara and Whyte [4]) Suppose X is a geodesic metric space. X is quasi-isometric to a simplicial tree if $H_1(X)$ is uniformly generated and X is of asymptotic dimension* 1.

Corollary 7.2 was used in [4] to show that finitely presented groups of asymptotic dimension 1 are virtually free (see also [5] and [8]).

8. Property A

Property A of G. Yu (see [15,10]) is usually defined for metric spaces of bounded geometry (that means the number of points in each r-ball $B(x, r)$ does not exceed $n(r) < \infty$ for each $r > 0$) as the condition that for each R, $\epsilon > 0$ there is $S > 0$ and finite subsets A_x of $X \times \mathbb{N}$, $x \in X$, so that $A_x \subset B(x, S) \times \mathbb{N}$ for each $x \in X$ and $\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \epsilon$ if $d(x, y) \leqslant R$. Here $A \Delta B := (A \setminus B) \cup (B \setminus A)$ is the *symmetric difference* of sets A and B.

For arbitrary metric spaces X one can use Condition 2 of Theorem 1.2.4 of [14]:

Definition 8.1. X has *Property A* if and only if for each R, $\epsilon > 0$ there is a function $\xi : X \to l^1(X)$ and $S > 0$ such that $\|\xi_x\|_1 = 1$ for each $x \in X$, $\|\xi_x - \xi_y\| < \epsilon$ if $d(x, y) \leqslant R$, and ξ_x is supported in $B(x, S)$ for each $x \in X$.

Notice one can always assume ξ_x has non-negative values (replace ξ_x by its absolute value). The conditions in Definition 8.1 are weaker than the original definition of Yu stated in the beginning of this section but both are equivalent for spaces of bounded geometry (see [14]).

In this section we redefine Property A of Yu in terms of coarse simplicial complexes. Given a simplicial complex K by $|K|_m$ we mean the geometric realization of K equipped with the metric resulting from considering $|K|$ as a subset of $l^1(K)$. It is obvious every simplicial map $f : K \to L$ induces a short map $f : |K|_m \to |L|_m$.

Given two functions $f, g : |K| \to |L|$ we say they are contiguous if for every simplex Δ of K there is a simplex Δ_1 of L such that $f(|\Delta|) \cup g(|\Delta|) \subset |\Delta_1|$. That generalizes the concept of contiguity between simplicial maps of simplicial complexes.

Definition 8.2. A coarse simplicial complex $\mathcal{K} = \{K_1 \to K_2 \to \cdots\}$ has *Property A* if for each $k \geqslant 1$ and each $\epsilon > 0$ there is $n > k$ and a function $f : |K_k|_m \to |K_n|_m$ such that f is contiguous to $i_{k,n} : |K_k| \to |K_n|$ and the diameter of $f(|\Delta|)$ is at most ϵ for each simplex Δ of K_k.

Theorem 8.3. *A metric space X has Property A if and only if its Rips complex has Property A.*

Proof. Suppose X has Property A as in Definition 8.1 and R, $\epsilon > 0$. There is a function $\xi : X \to l^1(X)$ and $S > R$ such that $\|\xi_x\|_1 = 1$ for each $x \in X$, ξ_x has non-negative values, $\|\xi_x - \xi_y\| < \epsilon$ if $d(x, y) \leqslant R$, and ξ_x is supported in $B(x, S)$ for each $x \in X$. By adjusting the values of ξ_x we may assume its support is finite for each $x \in X$ (pick a finite subset $C(x)$ so that $\sum_{y \in C(x)} \xi_x(y) > 1 - \epsilon/2$ and shift the sum of remaining values to a point in $C(x)$). Therefore ξ may be viewed as a function from vertices of $\mathrm{Rips}_R(X)$ to $|\mathrm{Rips}_{2S}(X)|_m$ and can be extended over $|\mathrm{Rips}_R(X)|_m$ (use the convex structure of $l^1(X)$ and extend the function from vertices over simplices via linear combinations) so that the resulting $\xi : |\mathrm{Rips}_R(X)|_m \to |\mathrm{Rips}_{2S+R}(X)|_m$ is contiguous to the inclusion-induced $|\mathrm{Rips}_R(X)|_m \to |\mathrm{Rips}_{2S+R}(X)|_m$ and $\xi(|\Delta|)$ is of diameter at most ϵ for any simplex Δ of $\mathrm{Rips}_R(X)$.

Conversely, if $\xi : |\mathrm{Rips}_R(X)|_m \to |\mathrm{Rips}_S(X)|_m$ is contiguous to the inclusion-induced $|\mathrm{Rips}_R(X)|_m \to |\mathrm{Rips}_S(X)|_m$ and $\xi(|\Delta|)$ is of diameter at most ϵ for any simplex Δ of $\mathrm{Rips}_R(X)$, then ξ restricted to vertices of $\mathrm{Rips}_R(X)$ gives a function $\mu : X \to l^1(X)$ by $\xi(x) = \sum_{y \in X} \mu_x(y) \cdot y$. Since for each $x \in X$ there is a simplex Δ_x of $\mathrm{Rips}_S(X)$ containing both x and $\xi(x)$, the carrier of μ_x is contained in $B(x, S)$. Also, if $d(x, y) < R$, $[x, y]$ forms a simplex in $\mathrm{Rips}_R(X)$ and its image is of diameter at most ϵ resulting in $\|\mu_x - \mu_y\| \leqslant \epsilon$. □

Since having Property A is an invariant of large scale equivalence of coarse simplicial complexes, Theorem 8.3 really says X has Property A if and only if any of its coarse simplicial complexes has Property A.

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